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ON NECESSARY OPTIMALITY CONDITIONS IN PROBLEMS OF CONTROLLING TRANSPORT PROCESSES

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A method given in [1] is used to establish the necessary conditions of optimality (in the form of the minimum principle) for a controlled unsteady process of transport of particles described by a general integro-differential kinetic equation. The control functions enter the initial conditions of the process, appearing in the inhomogeneous term of the equation (source), or in the absorption coefficient.

The problems of controlling the systems with distributed parameters connected with integro-differential equations are of importance from the theoretical point of view and in the practical applications such as the theory of control of nuclear reactors [3].

1. Let us consider a controlled process of transport of particles, connected with the mixed problem for an unsteady integro-differential transport equation

$$\frac{\partial}{\partial t}\psi(X, v, t) + (v, \nabla_x)\psi(X, v, t) + |v|\Sigma(X, v, t)\psi(X, v, t) = \tag{1.1}$$
$$\int_V |v'|\{\Sigma_f(X, v', t)\nu_f(X, v')K_f(X, v', v) +$$
$$\Sigma_s(X, v', t)K_s(X, v', v)\}\psi(X, v', t)\,dv' + q(X, v, t)$$
$$\psi(X, v, t)|_{t=+0} = g(X, v, u(X, v))$$
$$\psi(X, v, t)|_{X\in S} = 0, \quad (v\cdot n(X)) < 0, \quad t \in [0, T]$$
$$X = \{x_1, x_2, x_3\} \in G,$$
$$v = \{v_1, v_2, v_3\} \in V, \ |v| > \alpha = \text{const} > 0$$

Here G is a convex region in $E_3\ \{x_1, x_2, x_3\}$ bounded by a smooth surface S, $n = n(X)$ is the outward normal at the point $X \in S$, V is a bounded region belonging to $E_3\ \{v_1, v_2, v_3\}$ and $u(X, v)$, $(X, v) \in G \times V$ is a control vector function bounded and measurable in the given region, and assuming the values from $\Omega \subset E_m$ (we call such controls admissible).

In the following the operator $\partial / \partial t + (v, \nabla_x)$ will be denoted by $\partial / \partial l$ which is a derivative taken along the characteristic of this operator.

Let us pose the following optimal problem for (1.1): to find amongst the admissible controls $u(X, v)$ satisfying the restriction

$$L(u) = \int_D L(X, v, t, u(X, v), \psi_u(X, v, t))\,dD \leqslant 0 \tag{1.2}$$
$$D = G \times V \times [0, T], \qquad dD = dX\,dv\,dt$$

a control $u(X, v)$ which minimizes the functional

$$\Phi(u) = \int_G \int_V \Phi(X, v, u(X, v), \psi_u(X, v, T))\, dX\, dv \tag{1.3}$$

where T denotes a fixed instant.

We shall assume that the functions

$$q(X, u, t),\ g(X, v, u),\ \Sigma(X, v, t),\ K_f(X, w, v),\ K_s(X, w, v)$$

are measurable with respect to the variables (X, v, w, t), and $g(X, v, u)$ is continuous in u in the corresponding domains of definition. We shall also assume that the following conditions of boundedness and summability hold for any admissible control $u(X, v)$

$$0 \leqslant |v| \Sigma(X, v, t) \leqslant C_1, \qquad 0 \leqslant \nu_f(X, v) \leqslant C_2 \tag{1.4}$$

$$\max_G \int_V \left\{ \int_V |K_s(X, v', v)|^q dv' \right\}^{p/q} dv \leqslant C_3$$

$$\max_G \int_V \left\{ \int_V |K_f(X, v', v)|^q dv' \right\}^{p/q} dv \leqslant C_4$$

$$\frac{1}{p} + \frac{1}{q} = 1, \quad p > 1; \quad \|q(X, v, t)\|_{L_p(D)} \leqslant C_5$$

$$|g(X, v, u(X, v))| \leqslant A(X, v), \quad A(X, v) \in L_p(G \times V)$$

where $C_1 - C_5$ are constants.

In the functionals (1.2) and (1.3) we assume that $L(X, v, t, u, z)$ and $\Phi(X, v, u, z)$ are measurable in X, v and t, continuous in u and twice continuously differentiable with respect to z in the corresponding domains of definition and also increase in z according to the following power relations

$$|L|, |\Phi| \leqslant M_1 z^p; \quad |L_z'|, |\Phi_z'| \leqslant M_2 z^{p-1} \tag{1.5}$$

$$|L_{zz}''|, |\Phi_{zz}''| \leqslant M_3 z^{p-2}$$

where $M_1 \ldots - M_3$ are constants. These conditions find use in deriving the variations of the functionals.

2. Let $u^0(X, v)$ be the optimal control and $\psi_0(X, v, t)$ the corresponding L_p-solution of (1.1). We construct an impulsive variant $u^\varepsilon(X, v)$ of the control $u^0(X, v)$ as follows. We take a finite set of pairwise different points $(X_i, v_i) \in G \times V$. For each set of nonnegative numbers γ^{ik} such $\varepsilon_\gamma > 0$ can be found that for $0 < \varepsilon < \varepsilon_\gamma$, the rectangles

$$\Pi_{ik}^\varepsilon \equiv \Pi_{ik}^\varepsilon(X_i, v_i) = \Big\{(X, v)\colon x_{i1} - \varepsilon \sum_{l=1}^{k} \gamma^{il} < x_1 \leqslant$$

$$x_{i1} - \varepsilon \sum_{l=1}^{k-1} \gamma^{il}, \quad x_{is} - \varepsilon k < x_s \leqslant x_{is} - \varepsilon(k-1), \ s = 2, 3;$$

$$v_{in} - \varepsilon k < v_n \leqslant v_{in} - \varepsilon(k-1), \quad n = 1, 2, 3\Big\} \quad (|\Pi_{ik}^\varepsilon| = \varepsilon^6 \gamma^{ik})$$

do not intersect pairwise. Let $u = \{u_{ik}\}$ be a finite set of vectors belonging to Ω.

The variant $u^\varepsilon(X, v)$ with the parameters $\{(X_i, v_i)\}$, $u = \{u_{ik}\}$ and $\gamma = \{\gamma^{ik}\}$ is determined as follows:

$$u^\varepsilon(X, v) = \begin{cases} u_{ik}, & (X, v) \in \Pi_{ik}{}^\varepsilon \\ u^0(X, v), & (X, v) \in G \times V \setminus \bigcup\limits_{i,k} \Pi_{ik}{}^\varepsilon \end{cases}$$

We note that the mixed problem (1.1) with the conditions (1.4) is stable under the perturbation of control. (Similar variants can also be constructed for the cases when the control enters $q(X, v, t)$ and $\Sigma(X, v, t)$). We shall formulate this in the following theorem.

Theorem. For any variant $u^\varepsilon(X, v)$ there exists a positive number $\varepsilon^* \leqslant \varepsilon_\gamma$ such that each control $u^\varepsilon(X, v)$, $0 \leqslant \varepsilon \leqslant \varepsilon^*$ has, under the condition (1.4), a corresponding L_p- solution $\psi_\varepsilon(X, v, t)$ of the problem (1.1), unique in D.

The following inequality

$$\| \psi_\varepsilon(X, v, t) - \psi_0(X, v, t) \|_{L_p(D)} + \left\| \frac{\partial \psi_\varepsilon}{\partial l} - \frac{\partial \psi_0}{\partial l} \right\|_{L_p(D)} \leqslant$$
$$C \| g(X, v, u^\varepsilon(X, v)) - g(X, v, u^0(X, v)) \|_{L_p(G\times V)}$$

holds for this solution from which it follows that for $\varepsilon \to 0$

$$\| \psi_\varepsilon(X, v, t) - \psi_0(X, v, t) \|_{L_p(D)} + \left\| \frac{\partial \psi_\varepsilon}{\partial l} - \frac{\partial \psi_0}{\partial l} \right\|_{L_p(D)} \to 0$$

The above theorem follows from the theorems established in [4].

3. To find the first variations of Φ and L we must construct integral representations for the increments $\Delta_\varepsilon L = L(u^\varepsilon) - L(u^0)$ and $\Delta_\varepsilon \Phi = \Phi(u^\varepsilon) - \Phi(u^0)$. Consider the increment $\Delta_\varepsilon L$,

$$\Delta_\varepsilon L = \left\{ \int_D \frac{\partial L(\eta_\varepsilon)}{\partial z} [\psi_\varepsilon(X, v, t) - \psi_0(X, v, t)] \, dD \right\} +$$
$$\int_D \left\{ L(X, v, t, u^\varepsilon(X, v), \psi_0(X, v, t)) - L(X, v, t, u^0(X, v), \psi_0(X, v, t)) \right\} dD \equiv$$
$$\{\Delta_\varepsilon{}^1 L\} + \int_D \Delta_u L(X, v, t, u^\varepsilon, u^0, \psi_0) \, dD$$

$$\eta_\varepsilon = (X, v, t, u^\varepsilon(X, v), \psi_0(X, v, t) + \vartheta(\psi_\varepsilon - \psi_0)), \quad \varepsilon \geqslant 0, \quad 0 \leqslant \theta \leqslant 1$$

We denote by $A\psi$ a linear integro-differential operator of the form

$$A\psi = \frac{\partial \psi}{\partial l} + |v| \Sigma(X, v, t) \psi - \int_V |v'| \{ \Sigma_s(X, v', t) K_s(X, v', v) +$$
$$\Sigma_f(X, v', t) \nu_f(X, v') K_f(X, v', v) \} \psi(X, v', t) \, dv'$$

Let us introduce a linear normed space $L_p{}^A(D)$ connected with A, consisting of the functions $\psi(X, v, t) \in L_p(D)$ with a generalized derivative $\partial\psi/\partial l \in L_p(D)$, possessing a bounded norm of the form

$$\| \psi(X, v, t) \|_{L_p{}^A} = \| \psi(X, v, t) \|_{L_p(D)} + \left\| \frac{\partial \psi(X, v, t)}{\partial l} \right\|_{L_p(D)}$$

and such that

$$\psi(X, v, t)|_{t=+0} = \varphi(X, v), \quad \psi(x, v, t) = 0, \quad X \in S \quad (v \cdot n) < 0$$

By the Theorem the equation $A\psi = \theta$ has a unique solution in the space $L_p^A(D)$ for any function $\varphi(X, v) \in L_p(G \times V)$, and the following estimate holds:

$$\|\psi(X, v, t)\|_{L_p^A(D)} \leqslant C \|\varphi(X, v)\|_{L_p(G\times V)} \tag{3.1}$$

Since to each $\varphi(X, v) \in L_p(G \times V)$ there corresponds a unique $\psi(X, v, t) \in L_p^A(D)$, then an operator B, $\psi(X, v, t) = B\varphi(X, v)$ exists which acts from $L_p(G \times V)$ into $L_p^A(D)$ and is bounded by virtue of (3.1), i.e.

$$\|B\|_{L_p(G\times V)\to L_p^A(D)} \leqslant C$$

Clearly, B^* acts from $L_p^A(D)^*$ into $L_p(G \times V)^*$. We note that

$$\Delta_\varepsilon \psi = \psi_\varepsilon(X, v, t) - \psi_0(X, v, t) \in L_p^A(D)$$

and we can write

$$A(\Delta_\varepsilon\psi) = 0, \qquad \Delta_\varepsilon\psi = B\,\Delta_u g$$
$$\Delta_u g = g(X, v, u^\varepsilon(X, v)) - g(X, v, u^0(X, v)) \tag{3.2}$$

Next we shall use the following integral formula:

$$T_\varepsilon(\psi) = \int_D \frac{\partial L(\eta_\varepsilon)}{\partial z} \psi(X, v, t)\, dD$$

to obtain for each $\varepsilon \geqslant 0$ a linear bounded functional $T_\varepsilon = T_\varepsilon(\psi)$ on the functions $\psi(X, v, t) \in L_p^A(D)$. The functional obviously satisfies the condition

$$T_\varepsilon(\Delta_\varepsilon\psi) = \Delta_\varepsilon^1 L \tag{3.3}$$

Each functional T_ε, $\varepsilon \geqslant 0$ is an element of the space $L_p^A(D)^*$, and as such, is transformed by the operator B^* into a certain element of the space $L_p(G \times V)^*$. By (3.2), (3.3) and the Riesz theorem on representation of a linear functional according to which to each functional Q_ε on $L_p(G \times V)$ there corresponds a unique function χ_ε belonging to $L_q(G \times V)$, $1/p + 1/q = 1$ so that

$$Q_\varepsilon(z) = \int_{G\times V} \chi_\varepsilon(X, v)\, z(X, v)\, dX\, dv, \qquad z(X, v) \in L_p(G\times V)$$

we have

$$\Delta_\varepsilon^1 L = T_\varepsilon(\Delta_\varepsilon\psi) = (\tau_\varepsilon, \Delta_\varepsilon\psi) = (\tau_\varepsilon, B\Delta_u g) =$$
$$(B^*\tau_\varepsilon, \Delta_u g) = Q_\varepsilon(\Delta_u g) = \int_{G\times V} \chi_\varepsilon(X, \Delta_u g(X, v, u^\varepsilon, u^0)\, dX\, dv$$

From this we obtain the integral expression for $\Delta_\varepsilon L$ in the form

$$\Delta_\varepsilon L = \int_{G\times V} \chi_\varepsilon(X, v)\, \Delta_u g(X, v, u^\varepsilon, u^0)\, dX\, dv + \tag{3.4}$$
$$\int_D \Delta_u L(X, v, t, u^\varepsilon, u^0, \psi_0)\, dD$$
$$\chi_\varepsilon(X, v) = B^*\tau_\varepsilon, \quad T_\varepsilon(\psi) = (\tau_\varepsilon, \psi)$$

In the similar manner we find the increment $\Delta_\varepsilon \Phi$,

$$\Delta_\varepsilon \Phi = \int\limits_{G\times V} \nu_\varepsilon (X, v)\, \Delta_u g (X, v, u^\varepsilon, u^0)\, dX\, dv + \tag{3.5}$$

$$\int\limits_{G\times V} \Delta_u \Phi (X, v, u^\varepsilon, u^0, \psi_0)\, dX\, dv$$

$$\nu_\varepsilon (X, v) = B^*{}_T \rho_\varepsilon, \quad R_\varepsilon (\psi) = (\rho_\varepsilon, \psi)$$

4. The integral representations (3.4) and (3.5) make it possible to derive the first variations of the functionals L and Φ on any impulsive variant $u^\varepsilon (X, v)$ defined by the corresponding sets

$$\{(X_i, v_i)\}, \{u_{ik}\}, \{\gamma^{ik}\}, \quad 1 \leqslant i \leqslant N, \quad 1 \leqslant k \leqslant M \tag{4.1}$$

First, we find the variation δL,

$$\delta L = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^6} \Delta_\varepsilon L = \lim_{\varepsilon\to 0} \sum_{\substack{1\leqslant i\leqslant N\\ 1\leqslant k\leqslant M}} \gamma^{ik} \Big\{ \frac{1}{|\Pi_{ik}{}^\varepsilon|} \int\limits_{\Pi_{ik}{}^\varepsilon (X_i, v_i)} \chi_\varepsilon (X, v)$$

$$\Delta_u g (X, v, u^\varepsilon, u^0)\, dX\, dv +$$

$$\frac{1}{|\Pi_{ik}{}^\varepsilon|} \int\limits_{\Pi_{ik}{}^\varepsilon (X_i, v_i)} \int\limits_0^T \Delta_u L (X, v, t, u^\varepsilon, u^0, \psi_0)\, dD \Big\} =$$

$$\sum_{\substack{1\leqslant i\leqslant N\\ 1\leqslant k\leqslant M}} \gamma^{ik} \lim_{\varepsilon\to 0} \{\lambda_{ik}{}^\varepsilon (X_i, v_i) + \mu_{ik}{}^\varepsilon (X_i, v_i, T)\}$$

It can be shown that the following limiting passages can be realized almost everywhere in $G \times V$:

$$\lim_{\varepsilon\to 0} \lambda_{ik}{}^\varepsilon (X, v) = \lambda_{ik}{}^0 (X, v) = \chi_0 (X, v)\, \Delta_u g (X, v, u_{ik}, u^0)$$

$$\lim_{\varepsilon\to 0} \mu_{ik}{}^\varepsilon (X, v, T) = \mu_{ik}{}^0 (X, v, T) = \int\limits_0^T \Delta_u L (X, v, t, u_{ik}, u^0, \psi_0)\, dt \tag{4.2}$$

Thus for any pair of numbers N and M and any set

$$u = \{u_{ik}\}, \quad 1 \leqslant i \leqslant N, \quad 1 \leqslant k \leqslant M$$

there exists a set E_u^{NM}, $E_u^{NM} \subset G \times V$ with a full measure in $G \times V$, and a sequence $\varepsilon_s^{uNM} \to 0$, $s \to \infty$ such that the limiting passages (4.2) can be realized on any set $\{(X_i, v_i)\}$ belonging to E_u^{NM}. Therefore, if some variant $u^\varepsilon (X, v)$ has (4.1) and $\{(X_i, v_i)\} \subset E_u^{NM}$ as the defining sets, then the variation δL on this variant has the form

$$\delta L = \sum_{\substack{1\leqslant i\leqslant N\\ 1\leqslant k\leqslant M}} \gamma^{ik} \Big\{ \chi_0 (X_i, v_i)\, \Delta_u g (X_i, v_i, u_{ik}, u^0 (X_i, v_i)) +$$

$$\int\limits_0^T \Delta_u L (X_i, v_i, t, u_{ik}, u^0 (X_i, v_i), \psi_0 (X_i, v_i, t))\, dt \Big\}$$

Carrying out the usual argumentation we can derive the variation on a wider class of variants defined by the following finite sets:

$$\{(X_i, v_i)\}, \quad u = \{u_{ik}\}, \quad \gamma = \{\gamma^{ik}\}, \quad u_{ik} \in \Omega_1, \quad \gamma^{ik} \geqslant 0 \tag{4.3}$$

with $(X_i, v_i) \in E_L$ (Ω_1 is a denumerable vector net dense everywhere in Ω) where we obtain the variation in the form

$$\delta L = \sum_{i,k} \gamma^{ik} \Big\{ \chi_0 (X_i, v_i)\, \Delta_u g (X_i, v_i, u_{ik}, u^0 (X_i, v_i)) + \int_0^T \Delta_u L (X_i, v_i, t, u_{ik}, u^0 (X_i, v_i), \psi_0 (X_i, v_i, t))\, dt \Big\} \tag{4.4}$$

Similarly, for the functional $\Phi(u)$ we can introduce a set $E_\Phi \subset G \times V$, $\operatorname{mes} E_\Phi = \operatorname{mes} (G \times V)$ such that on an arbitrary variant $u^\varepsilon (X, v)$ defined by the finite sets (4.3) with $(X_i, v_i) \subset E_\Phi$, the first variation $\delta\Phi$ has the form

$$\delta\Phi = \sum_{i,k} \gamma^{ik} \{ \nu_0 (X_i, v_i)\, \Delta_u g (X_i, v_i, u_{ik}, u^0 (X_i, v_i)) + \Delta_u \Phi (X_i, v_i, u_{ik}, u^0 (X_i, v_i), \psi_0 (X_i, v_i, T)) \} \tag{4.5}$$

5. Using the formulas (4.4) and (4.5) we now obtain the necessary conditions of optimality in the form of a minimum principle. To each set of parameters (4.3) with $(X_i, v_i) \in E = E_L \cap E_\Phi$ there corresponds, in accordance with (4.4) and (4.5), a definite pair of variations δL and $\delta\Phi$ which can be considered as a vector belonging to the space E_2. The whole collection of such sets has a corresponding set K of vectors $\{\delta L, \delta\Phi\}$. Investigation of the structure of the set $K \subset E_2$ shows that K is a convex cone in E_2 with its apex at the coordinate origin and, that the cone does not intersect the open negative quadrant $R = \{x_1, x_2;\ x_1 < 0,\ x_2 < 0\}$. From this it follows that the convex nonintersecting cones K and R, diverge in E_2 on a certain straight line defined by the vector of the normal $\mu = \{\mu_1, \mu_2\} \in E_2$ directed towards the cone K, and the scalar product of μ and any vector belonging to K is nonnegative

$$\mu_1 \delta L + \mu_2 \delta\Phi \geqslant 0 \tag{5.1}$$

The above relation together with (4.4) and (4.5) makes it possible to obtain the necessary conditions of optimality in the form of a minimum principle. To do this, we must construct a specific set of parameters

$$(X_1, v_1),\ \{u_{11}\},\ \{\gamma^{11} = 1\}$$
$$u_{11} \in \Omega_1,\quad (X_1, v_1) \in E,\quad E \subset G \times V$$

Evaluating the corresponding set of variations and using (5.1), we obtain the necessary condition of optimality in the form of the following minimum principle.

The minimum principle. If $u^0 (X, v)$ is the optimal control and $\Psi_0 (X, v, t)$ the corresponding L_p-solution of the problem (1.1), then there exist functions $\chi_0 (X, v)$ and $\nu_0 (X, v)$ belonging to $L_q (G \times V)$ and defined uniquely by the operator relations

$$\chi_0 (X, v) = B^* \tau_0, \qquad \nu_0 (X, v) = B_T^* \rho_0 \tag{5.2}$$

and nonnegative numbers μ_1, μ_2 $(\mu_1^2 + \mu_2^2 \neq 0)$ such that the relation

$$\Lambda (X, v, t, u^0 (X, v)) \equiv \tag{5.3}$$
$$\mu_1 \Big\{ \chi_0 (X, v)\, g (X, v, u^0) + \int_0^T L (X, v, t, u^0, \Psi_0 (X, v, t))\, dt \Big\} +$$

$$\mu_2 \{\nu_0(X, v) g(X, v, u^0) + \Phi(X, v, u^0, \Psi_0(X, v, T))\} = \inf_{u \in \Omega} \Lambda(X, v, t, u(X, t))$$

holds almost everywhere in $G \times V$.

We note that the functions $\nu_0(X, v)$ and $\chi_0(X, v)$ defined in (5.2) represent the solutions of certain conjugate problems. Let $\varphi_1(X, v, t)$ be a solution of the following mixed problem:

$$A^*\varphi_1 \equiv -\frac{\partial}{\partial t}\varphi_1(X, v, t) - (v, \nabla_x)\varphi_1(X, v, t) + \tag{5.4}$$
$$|v|\Sigma(X, v, t)\varphi_1 - \int_V |v|\{\Sigma_f(X, v, t)\nu_f(X, v)K_f(X, v, v') +$$
$$\Sigma_s(X, v, t)K_s(X, v, v')\}\varphi_1(X, v', t)\,dv' = \frac{\partial L(\eta_0)}{\partial z}$$
$$\varphi_1(X, v, t)|_{t=T-0} = 0$$
$$\varphi_1(X, v, t)|_{X \in S} = 0, \qquad (n(X)\cdot v) > 0$$

Then we can easily see that $\chi_0(X, v) = \varphi_1(X, v, 0)$.

Similarly, if $\varphi_2(X, v, t)$ is a solution of the mixed problem

$$A^*\varphi_2(X, v, t) = 0, \quad \varphi_2(X, v, t)|_{t=T-0} = \frac{\partial\Phi(\eta_0)}{\partial z} \tag{5.5}$$
$$\varphi_2(X, v, t)_{X \in S} = 0, \qquad (n(X)\cdot v) > 0$$

then

$$\nu_0(X, v) = \varphi_2(X, v, 0)$$

Furthermore we note that $\chi_0(X, v)$ and $\nu_0(X, v)$ can also be written in the form

$$\chi_0(X, v) = \int_D \frac{\partial L(\eta_0(t, Y, w))}{\partial z} G_0(0, X, v \mid t, Y, w)\,dD$$
$$\nu_0(X, v) = \int_{G \times V} \frac{\partial\Phi(\eta_0(T, Y, w))}{\partial z} G_0(0, X, v \mid T, Y, w)\,dY\,dw$$

Here $G_0(s, Y, w \mid t, X, v)$ is the Green's function of the mixed problem which corresponds to the optimal control $u^0(X, v)$ and defined as follows:

$$AG_0(s, Y, w \mid t, X, v) = 0 \quad \text{(in the variables } t, X, v)$$
$$A^*G_0(s, Y, w \mid t, X, v) = 0 \quad \text{(in the variables } s, Y, w)$$

where $G_0(s, Y, w \mid t, X, v) \to \delta(Y - X)\,\delta(w - v)$ when $(t - s) \downarrow 0$ (it is assumed that $\Sigma_f(X, v, t)$, $\Sigma_s(X, v, t)$ and $\Sigma(X, v, t)$ all vanish identically outside G).

6. We deal in the same manner with the problem of controlling a transport process in the case when the control vector function $u(X, v, t)$ enters the coefficient of absorption $\Sigma(X, v, t, u)$ and the source $q(X, v, t, u)$, but is not contained in the boundary condition of the problem (1.1) (problem A).

Let us pose for the problem A the following optimal problem: to find among the admissible controls $u(X, v, t)$ (i.e. $u(X, v, t)$, $(X, v, t) \in D$ is a vector control function, measurable and bounded within the stated region and assuming the values from the bounded set $\Omega \subset E_m$), satisfying the restriction

$$N(u) = \int_D N(X, v, t, u(X, v, t), \psi_u(X, v, t))\, dD \leqslant 0$$

the control which minimizes the functional

$$J(u) = \int_D F(X, v, t, u(X, v, t), \psi_u(X, v, t))\, dD$$

Let $N(X, v, t, u, z)$ and $F(X, v, t, u, z)$ satisfy the same conditions as $L(X, v, t, u, z)$ (see Sect. 1). Let the conditions (1.4) also hold and let

$$\begin{aligned} &0 \leqslant |v| \Sigma(X, v, t, u(X, v, t)) \leqslant C_1 \\ &|q(X, v, t, u(X, v, t))| \leqslant B(X, v, t) \in L_p(D)| \end{aligned}$$

for any admissible control $u(X, v, t)$. Finally, let Σ and q be continuous in u. Then we have the following minimum principle.

The minimum principle. If $u^0(X, v, t)$ is the optimal control and $\psi_0(X, v, t)$ the corresponding L_p-solution of the problem A, then there exist functions $\alpha_0(X, v, t)$ and $\beta_0(X, v, t)$ belonging to $L_q(D)$ and defined uniquely by the operator relations

$$\alpha_0(X, v, t) = (A_0^{-1})^* \tau_0, \qquad \beta_0(X, v, t) = (A_0^{-1})^* \rho_0$$

and nonnegative numbers μ_1 and μ_2 $(\mu_1^2 + \mu_2^2 \neq 0)$ such that the relation

$$\begin{aligned} \Lambda_A(X, v, t, u^0(X, v, t)) \equiv \mu_1 \{&\alpha_0(X, v, t)[q(X, v, t, u^0) - \\ &\Psi_0(X, v, t)\, |v|\, \Sigma(X, v, t, u^0)] + F(X, v, t, u^0, \Psi_0(X, v, t))\} + \\ \mu_2 \{&\beta_0(X, v, t)[q(X, v, t, u^0) - \psi_0(X, v, t)\, |v|\, \Sigma(X, v, t, u^0)] + \\ &N(X, v, t, u^0, \Psi_0(X, v, t))\} = \inf_{u \in \Omega} \Lambda_A(X, v, t, u(x, v, t)) \end{aligned}$$

holds almost everywhere in D. In the above expression τ_0 and ρ_0 are functionals uniquely defined by the functionals I and N and by the optimal pair Ψ_0, u^0.

Let $G_0(s, Y, w \mid t, X, v)$ denote the Green's function corresponding to the optimal control $u^0(X, v, t)$. Then

$$\alpha_0(X, v, t) = \int_t^T \int_{G \times V} \frac{\partial F(\eta_0(s, Y, w))}{\partial z} G_0(t, X, v \mid s, Y, w)\, dD$$

$$\beta_0(X, v, t) = \int_t^T \int_{G \times V} \frac{\partial N(\eta_0(s, Y, w))}{\partial z} G_0(t, X, v \mid s, Y, w)\, dD$$

We note, in addition, that $\alpha_0(X, v, t)$ and $\beta_0(X, v, t)$ are solutions of the conjugate problems analogous to (5.5) and (5.4).

Methods similar to those given in [5, 6] can be used to compute the Green's function, and an analogous minimum principle can be considered for the stationary problems of the transport theory.

7. The minimum principles obtained in Sect. 5 and 6 reduce the problem of determining optimal controls to finding the minima of the corresponding functionals and to

solving the transport equation by approximate methods. Methods described in [7 – 11] can be used to solve the numerical problems.

As an example, we consider the following problem connected with the transport equation

$$\frac{\partial}{\partial t}\psi_u(z,\mu,t)+\mu\frac{\partial}{\partial z}\psi_u(z,\mu,t)+\sigma\psi_u=\frac{c}{2}\int_{-1}^{1}\psi_u(z,\mu',t)\,d\mu' \tag{7.1}$$

$$\psi_u(z,\mu,t)|_{t=+0}=g(z,\mu)\equiv u(z,\mu)$$

$$\psi_u(0,\mu,t)=0,\ \mu>0;\ \psi_u(a,\mu,t)=0,\ \mu<0$$

Let the function $\chi(z,\mu)\in L_2\{(0,a)\times(-1,1)\}$ be specified. We seek such an admissible control $u(z,\mu)$ that the generalized L_2-solution of the problem (7.1) minimizes the functional

$$\Phi(u)=\int_{-1}^{+1}\int_{0}^{a}|\psi_u(z,\mu,T)-\chi(z,\mu)|^2\,dz\,d\mu$$

Using formula (5.3) we write the necessary conditions of optimality of the control in the form of the following minimum principle:

$$\nu_0(z,\mu)\,u^0(z,\mu)+\Phi(z,\mu,u^0(z,\mu),\psi_0(z,\mu,T))=\inf_{u\in\Omega}\{\nu_0(z,\mu)\,u+\Phi(z,\mu,u,\psi_0(z,\mu,T))\}$$

Thus, it is necessary to find a control $u(z,\mu)$ minimizing the functional

$$I(u)=\nu_0(z,\mu)\,u(z,\mu)+\Phi(u,\Psi_u(z,\mu,T))$$

Here $\nu_0(z,\mu)$ is the solution of a certain conjugate problem analogous to (5.5). Namely, if $\varphi_u(z,\mu,t)$ represents a generalized L_2-solution of the problem

$$-\frac{\partial\varphi_u(z,\mu,t)}{\partial t}-\mu\frac{\partial\varphi_u(z,\mu,t)}{\partial z}+\sigma\varphi_u=\frac{c}{2}\int_{-1}^{+1}\varphi_u(z,\mu',t)\,d\mu' \tag{7.2}$$

$$\varphi_u(z,\mu,t)|_{t=T-0}=2\{\psi_u(z,\mu,T)-\chi(z,\mu)\}$$

$$\varphi_u(0,\mu,t)=0,\ \mu<0;\ \varphi_u(a,\mu,t)=0,\ \mu>0$$

then

$$\nu_0(z,\mu)=\varphi_0(z,\mu,0)$$

We note that in the above example the control function $u(z,\mu)$ is not, generally speaking, restricted in any way and $\Phi(u)$ does not depend explicitly on the control $u(z,\mu)$. For this reason the functional $I(u)$ is a linear function of the control $u(z,\mu)$ and attains a minimum only for $\nu_0(z,\mu)=0$ or in other words, $\varphi_0(z,\mu,0)=0$. The latter together with (7.2) yield the relation $\psi_0(z,\mu,T)=\chi(z,\mu)$. This in fact determines the optimal control $u^0(z,\mu)$, which can be any control with the property that the corresponding solution satisfies the condition $\psi_u(z,\mu,T)=\chi(z,\mu)$ at the instant $t=T$.

The figures illustrate the controls corresponding to various standard values of $\chi(z,\mu)$ and the solutions of (7.1) at the instant $t=T$ obtained according to the controls computed, and this fully agrees with the arguments presented above. Figures 1 – 3 correspond to

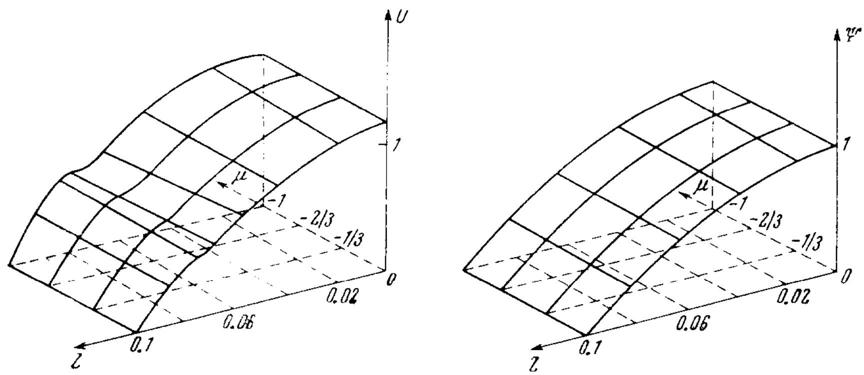

Fig. 1

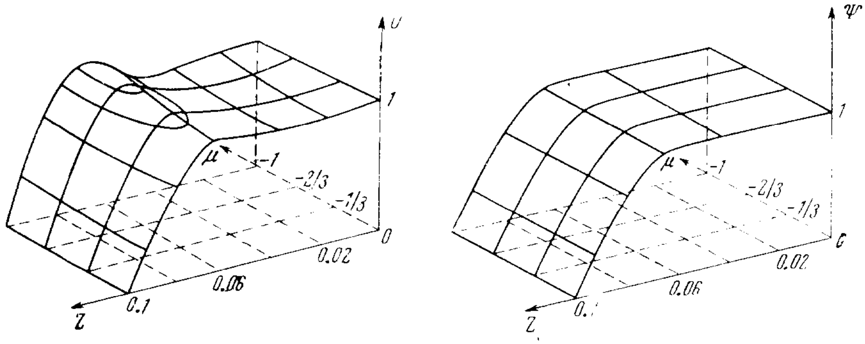

Fig. 2

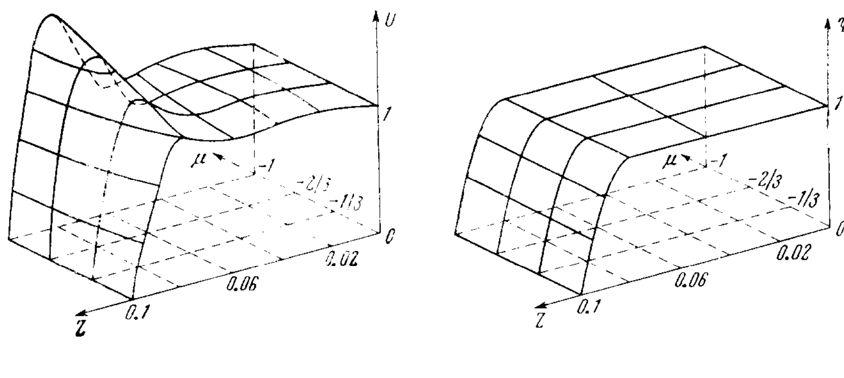

Fig. 3

$$\chi_1(z,\mu)=\begin{cases}\cos\dfrac{\pi}{0.2}z, & z\in[0,0.1], \quad \mu<0\\ \sin\dfrac{\pi}{0.2}z, & z\in[0,0.1], \quad \mu>0\end{cases}$$

$$\chi_2(z,\mu)=\begin{cases}1, & z\in[0,0.08], \quad \mu<0\\ \cos\dfrac{\pi}{0.04}(z-0.08) & z\in[0.08,0.1], \quad \mu<0\\ 1, & z\in[0.02,0.1], \quad \mu>0\\ \sin\dfrac{\pi}{0.04}z, & z\in[0,0.02], \quad \mu>0\end{cases}$$

$$\chi_3(z,\mu)=\begin{cases}1, & z\in[0,0.09], \quad \mu<0\\ \cos\dfrac{\pi}{0.02}(z-0.09) & z\in[0.09,0.1], \quad \mu<0\\ 1, & z\in[0.01,0.1], \quad \mu>0\\ \sin\dfrac{\pi}{0.02}z, & z\in[0,0.01], \quad \mu>0\end{cases}$$

The transport equation was solved by an apptoximate method given in [10] and the functional was minimized using the method of coordinate descent [12].

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